Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 323 (2006) 456–472
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Local and global existence and uniqueness results for impulsive functional differential equations with multiple delay

Abdelghani Ouahab

Laboratoire de Mathématiques, Université de Sidi Bel Abbès BP 89, 22000 Sidi Bel Abbès, Algeria

Received 10 February 2005
Available online 28 November 2005
Submitted by J. Henderson

Abstract

In this paper, we discuss local and global existence and uniqueness results for first-order impulsive functional differential equations with multiple delay. We shall rely on a fixed point theorem of Schaefer and a nonlinear alternative of Leray–Schauder. For the global existence and uniqueness we apply a recent nonlinear alternative of Leray–Schauder type in Fréchet spaces, due to Frigon and Granas [M. Frigon, A. Granas, Résultats de type Leray–Schauder pour des contractions sur des espaces de Fréchet, Ann. Sci. Math. Québec 22 (2) (1998) 161–168].
© 2005 Elsevier Inc. All rights reserved.

Keywords: Impulsive functional differential equations; Uniqueness; Multiple delay; Fréchet spaces

1. Introduction

This paper is concerned with the existence and uniqueness of solutions, for first-order functional differential equations with impulsive effects and multiple delay. In Section 3, we will consider local existence and uniqueness results for first-order impulsive functional differential equations with fixed moments and multiple delay

$$y'(t) = f(t, y_t) + \sum_{i=1}^{n_*} y(t - T_i) \quad \text{a.e. } t \in J := [0, b] \backslash \{t_1, t_2, \dots, t_m\}, \tag{1}$$

E-mail address: ouahab@univ-sba.dz.

0022-247X/$ – see front matter © 2005 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2005.10.015

$$y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{2}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \tag{3}$$

where $n_* \in \{1, 2, \dots\}$, $r = \max_{1 \leqslant i \leqslant n_*} T_i$, $f : J \times \mathcal{D} \to \mathbb{R}^n$ is a given function, $\mathcal{D} = \{\psi : [-r, 0] \to \mathbb{R};\ \psi$ is continuous everywhere except for a finite number of points $\bar{t}$ at which $\psi(\bar{t})$ and $\psi(\bar{t}^+)$ exist and $\psi(\bar{t}^-) = \psi(\bar{t})\}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, $I_k \in C(\mathbb{R}^n, \mathbb{R}^n)$, $k = 1, 2, \dots, m$, are given functions satisfying some assumptions that will be specified later.

For any function y defined on $[-r, b]$ and any $t \in J$ we denote by y_t the element of $\mathcal{D}$ defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0].$$

Here $y_t(\cdot)$ represents the history of the state from time $t - r$, up to the present time t.

Impulsive differential equations have become more important in recent years in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. There has been a significant development in impulse theory, in recent years, especially in the area of impulsive differential equations with fixed moments; see the monographs of Bainov and Simeonov [2], Lakshmikantham et al. [12], Samoilenko and Perestyuk [14] and the references therein. The main theorems of this paper extends the problem (1)–(3) considered by Benchohra et al. [3] when the impulse times are constant. Our approach is based on Schaefer's fixed point theorem (see [15, p. 29]), the nonlinear alternative of Leray–Schauder type [6], the Banach fixed point theorem and a recent nonlinear alternative of Leray–Schauder type in Fréchet spaces, due to Frigon and Granas [7]. These results can be considered as a contribution to the literature.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

By $C(J, \mathbb{R}^n)$ we denote the Banach space of all continuous functions from J into $\mathbb{R}^n$ with the norm

$$\|y\|_\infty := \sup\{|y(t)|:\ t \in J\}.$$

Also $\mathcal{D}$ is endowed with norm $\|\cdot\|$ defined,

$$\|\phi\|_{\mathcal{D}} := \sup\{|\phi(\theta)|:\ -r \leqslant \theta \leqslant 0\},$$

$$L^1(J, \mathbb{R}^n) = \{y : J \to \mathbb{R}^n:\ y \text{ is Lebesgue integrable}\}.$$

Then, we are able to define

$$\|y\|_{L^1} = \int_0^b |y(t)|\, dt \quad \text{for } J := [0, b].$$

Let us add that two functions $y_1, y_2 : J \to \mathbb{R}^n$ such that the set $\{y_1(t) \neq y_2(t) \mid t \in J\}$ has Lebesgue measure equal to zero are considered as equal. It is well known that

$$(L^1(J, \mathbb{R}^n), \|\cdot\|_{L^1})$$

is a Banach space.

$AC^i(J, \mathbb{R}^n)$ is the space of functions $y : J \to \mathbb{R}^n$ i-differentiable in whose ith derivative, $y^{(i)}$, is absolutely continuous.

Definition 2.1. A map $f : J \times \mathcal{D} \to \mathbb{R}^n$ is said to be L^1-Carathéodory if

(i) $t \mapsto f(t,u)$ is measurable for each $u \in \mathcal{D}$;
(ii) $u \mapsto f(t,u)$ is continuous for almost all $t \in J$;
(iii) For each $q > 0$, there exists $h_q \in L^1(J, \mathbb{R}_+)$ such that

$$\left|f(t,u)\right| \leqslant h_q(t) \quad \text{for all } \|u\| \leqslant q \text{ and for almost all } t \in J.$$

3. Existence and uniqueness results

In order to define the solution of (1)–(3) we shall consider the space

$$PC = \big\{y : [0,b] \to \mathbb{R}^n\colon\ y\big(t_k^-\big) \text{ and } y\big(t_k^+\big) \text{ exist with } y\big(t_k^-\big) = y(t_k),\ k = 1,\dots,m \\ \text{and } y \in C\big([t_k, t_{k+1}), \mathbb{R}^n\big),\ k = 0,\dots,m\big\},$$

which is a Banach space with the norm

$$\|y\|_{PC} = \max\big\{\|y_k\|_{J_k},\ k = 0,\dots,m\big\},$$

where y_k is the restriction of y to $J_k = [t_k, t_{k+1}]$, $k = 0,\dots,m$. Let

$$\Omega = \big\{y : [-r,b] \to \mathbb{R}^n\colon\ y \in \mathcal{D} \cap PC\big\},$$

which is a Banach space with the norm

$$\|y\|_\Omega = \sup\big\{\big|y(t)\big|\colon\ t \in [-r,b]\big\}, \quad y \in \Omega.$$

Let us start by defining what we mean by a solution of problem (1)–(3).

Definition 3.1. A function $y \in \Omega \cap AC(J, \mathbb{R}^n)$, is said to be a solution of (1)–(3) if y satisfies the equation $y'(t) = f(t, y_t) + \sum_{i=1}^{n_*} y(t - T_i)$ a.e. on J, $t \neq t_k$, $k = 1,\dots,m$ and the conditions $y(t_k^+) - y(t_k^-) = I_k(y(t_k^-))$, $k = 1,\dots,m$ and $y(t) = \phi(t)$ on $[-r, 0]$.

We need the following auxiliary result.

Lemma 3.2. *Let $f : \mathcal{D} \to \mathbb{R}^n$ be a continuous function. Then y is the unique solution of the initial value problem*

$$y'(t) = f(y_t) + \sum_{i=1}^{n_*} y(t - T_i), \quad t \in J := [0,b],\ t \neq t_k,\ k = 1,\dots,m, \tag{4}$$

$$y\big(t_k^+\big) - y(t_k) = I_k\big(y\big(t_k^-\big)\big), \quad k = 1,\dots,m, \tag{5}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \tag{6}$$

where $r = \max_{1 \leqslant i \leqslant n_*} T_i$ if and only if y is a solution of impulsive integral functional differential equation

$$y(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0], \\ \phi(0) + \sum_{i=1}^{n_*} \int_{-T_i}^0 \phi(s)\,ds + \int_0^t f(y_s)\,ds \\ \quad + \sum_{i=1}^{n_*} \int_0^{t - T_i} y(s)\,ds + \sum_{0 < t_k < t} I_k(y(t_k^-)), & \text{if } t \in [0,b]. \end{cases} \tag{7}$$

Proof. Let y be a possible solution of the problem (4)–(6). Then $y|_{[-r,t_1]}$ is a solution to

$$y'(t) = f(y_t) + \sum_{i=1}^{n_*} y(t - T_i) \quad \text{for } t \in [0, b].$$

Assume that $t_k < t \leqslant t_{k+1}$, $k = 1, \dots, m$. By integration of above inequality yields

$$y\big(t_1^-\big) - y(0) = \int_0^{t_1} f(y_s)\,ds + \sum_{i=1}^{n_*} \int_0^{t_1} y(s - T_i)\,ds,$$

$$y\big(t_1^-\big) - y(0) = \int_0^{t_1} f(y_s)\,ds + \sum_{i=1}^{n_*} \int_{-T_i}^{t_1-T_i} y(s)\,ds,$$

$$y\big(t_2^-\big) - y\big(t_1^+\big) = \int_{t_1}^{t_2} f(y_s)\,ds + \sum_{i=1}^{n_*} \int_{t_1}^{t_2} y(s - T_i)\,ds,$$

$$y\big(t_2^-\big) - y\big(t_1^-\big) = I_1\big(y\big(t_1^-\big)\big) + \int_{t_1}^{t_2} f(y_s)\,ds + \sum_{i=1}^{n_*} \int_{t_1-T_i}^{t_2-T_i} y(s)\,ds,$$

$$\vdots$$

$$y\big(t_k^-\big) - y\big(t_{k-1}^+\big) = \int_{t_{k-1}}^{t_k} f(y_s)\,ds + \sum_{i=1}^{n_*} \int_{t_{k-1}}^{t_k} y(s - T_i)\,ds,$$

$$y\big(t_k^-\big) - y\big(t_{k-1}^-\big) = I_k\big(y\big(t_k^-\big)\big) + \int_{t_{k-1}}^{t_k} f(y_s)\,ds + \sum_{i=1}^{n_*} \int_{t_{k-1-T_i}}^{t_k-T_i} y(s - T_i)\,ds,$$

$$y(t) - y\big(t_k^-\big) = I_k\big(y\big(t_k^-\big)\big) + \int_{t_k}^{t} f(y_s)\,ds + \sum_{i=1}^{n_*} \int_{t_k-T_i}^{t-T_i} y(s)\,ds.$$

Then

$$y(t_1) - y(0) = \int_0^{t_1} f(y_s)\,ds + \sum_{i=1}^{n_*} \int_{-T_i}^{t_1-T_i} y(s)\,ds,$$

$$y(t_2) - y\big(t_1^-\big) = I_1\big(y\big(t_1^-\big)\big) + \int_{t_1}^{t_2} f(y_s)\,ds + \sum_{i=1}^{n_*} \int_{t_1-T_i}^{t_2-T_i} y(s)\,ds,$$

$$\vdots$$

$$y\big(t_k^-\big) - y(t_{k-1}) = I_k\big(y\big(t_k^-\big)\big) + \int_{t_{k-1}}^{t_k} f(y_s)\,ds + \sum_{i=1}^{n_*} \int_{t_{k-1-T_i}}^{t_k-T_i} y(s - T_i)\,ds,$$

$$y(t) - y\big(t_k^-\big) = I_k\big(y\big(t_k^-\big)\big) + \int_{t_k}^{t} f(y_s)\,ds + \sum_{i=1}^{n_*} \int_{t_k - T_i}^{t - T_i} y(s)\,ds.$$

Adding these together, we get

$$\begin{aligned} y(t) &= y(0) + \sum_{0<t_k<t} I_k\big(y\big(t_k^-\big)\big) + \int_0^t f(y_s)\,ds + \sum_{i=1}^{n_*} \int_{-T_i}^{t-T_i} y(s)\,ds \\ &= \phi(0) + \sum_{0<t_k<t} I_k\big(y\big(t_k^-\big)\big) + \int_0^t f(y_s)\,ds + \sum_{i=1}^{n_*} \int_0^{t-T_i} y(s)\,ds + \sum_{i=1}^{n_*} \int_{-T_i}^{0} y(s)\,ds. \end{aligned}$$

Thus

$$y(t) = \phi(0) + \sum_{0<t_k<t} I_k\big(y\big(t_k^-\big)\big) + \int_0^t f(y_s)\,ds + \sum_{i=1}^{n_*} \int_0^{t-T_i} y(s)\,ds + \sum_{i=1}^{n_*} \int_{-T_i}^{0} \phi(s)\,ds.$$

If y satisfies the integral equation (7), then y is solution of the problem (4)–(6). Let $t \in [0, b]\setminus \{t_1, \dots, t_m\}$ and

$$y(t) = \phi(0) + \sum_{0<t_k<t} I_k\big(y\big(t_k^-\big)\big) + \int_0^t f(y_s)\,ds + \sum_{i=1}^{n_*} \int_0^{t-T_i} y(s)\,ds + \sum_{i=1}^{n_*} \int_{-T_i}^{0} \phi(s)\,ds.$$

Hence

$$y'(t) = f(y_t) + \sum_{i=1}^{n_*} y(t - T_i).$$

We can easily prove that $y(t_k^+) - y(t_k^-) = I_k(y(t_k^-))$, $k = 1, \dots, m$. $\square$

We are now in a position to state and prove our existence result for the problem (1)–(3). For the study of this problem we first list the following hypotheses:

(H1) $F : J \times \mathcal{D} \to \mathbb{R}^n$ is an L^1-Carathéodory function;

(H2) There exist positive constants c_k, $k = 1, \dots, m$ such that

$$\big|I_k(y)\big| \leqslant c_k \quad \text{for all } y \in \mathbb{R}^n;$$

(H3) There exist a function $p \in L^1(J, \mathbb{R}_+)$ and a continuous nondecreasing function $\psi : [0, \infty) \to [0, \infty)$ such that

$$\big|f(t, x)\big| \leqslant p(t)\psi\big(\|x\|_{\mathcal{D}}\big) \quad \text{for a.e. } t \in J \text{ and each } x \in \mathcal{D}$$

with

$$\int_0^b m(s)\,ds < \int_c^{\infty} \frac{du}{u + \psi(u)},$$

where $c = \|\phi\|_{\mathcal{D}} + rn_*\|\phi\|_{\mathcal{D}} + \sum_{k=1}^m c_k$ and $m(t) = \max(n_*, p(t))$.

Theorem 3.3. *Assume that hypotheses* (H1)–(H3) *hold. Then the IVP* (1)–(3) *has at least one solution on* $[-r, b]$.

Proof. Transform the problem (1)–(3) into a fixed point problem. Consider the operator $N : \Omega \to \Omega$ defined by

$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [-r, 0], \\ \phi(0) + \sum_{i=1}^{n_*} \int_{-T_i}^{0} \phi(s)\, ds + \int_0^t f(s, y_s)\, ds \\ \quad + \sum_{i=1}^{n_*} \int_0^{t-T_i} y(s)\, ds + \sum_{0<t_k<t} I_k(y(t_k^-)), & \text{if } t \in [0, b]. \end{cases}$$

We shall show that the operator N is completely continuous.

Step 1. *N is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in Ω.
Then

$$\begin{aligned} \left|N(y_n)(t) - N(y)(t)\right| &\leqslant \int_0^t \left|f(s, y_{ns}) - f(s, y_s)\right| ds + n_* \int_0^t \left|y_n(s) - y(s)\right| ds \\ &\quad + \sum_{k=1}^{m} \left|I_k\big(y_n\big(t_k^-\big)\big) - I_k\big(y\big(t_k^-\big)\big)\right| \\ &\leqslant \int_0^b \left|f(s, y_{ns}) - f(s, y_s)\right| ds + n_* \int_0^t \left|y_n(s) - y(s)\right| ds \\ &\quad + \sum_{k=1}^{m} \left|I_k\big(y_n\big(t_k^-\big)\big) - I_k\big(y\big(t_k^-\big)\big)\right|. \end{aligned}$$

Since f is an L^1-Carathéodory function and I_k are continuous functions, we have by the Lebesgue dominated convergence theorem that

$$\begin{aligned} \left\|N(y_n) - N(y)\right\|_{\Omega} &\leqslant \left\|f(\cdot, y_n) - f(\cdot, y)\right\|_{L^1} + n_* b \|y_n - y\|_{\Omega} \\ &\quad + \sum_{k=1}^{m} \left|I_k\big(y_n\big(t_k^-\big)\big) - I_k\big(y\big(t_k^-\big)\big)\right| \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Step 2. *N maps bounded sets into bounded sets in* Ω.

Indeed, it is enough to show that for any $q > 0$ there exists a positive constant ℓ such that for each $y \in B_q = \{y \in \Omega\colon \|y\|_{\Omega} \leqslant q\}$ we have $\|N(y)\|_{\Omega} \leqslant \ell$. By Definition 3.1(iii) we have for each $t \in [0, b]$

$$\left|N(y)(t)\right| \leqslant \left|\phi(0)\right| + \sum_{k=1}^{n_*} \int_0^{T_i} \left|\phi(-s)\right| ds + \int_0^t \left|f(s, y_s)\right| ds$$

$$+\sum_{k=1}^{n_*}\int_0^{t-T_i}\left|y(s)\right|ds+\sum_{k=1}^{m}\left|I_k\left(y\left(t_k^-\right)\right)\right|$$

$$\leqslant \|\phi\|_{\mathcal{D}}+n_*r\|\phi\|_{\mathcal{D}}+\|h_q\|_{L^1}+n_*qb+\sum_{k=1}^{m}c_k.$$

Thus

$$\left\|N(y)\right\|_{\Omega}\leqslant \|\phi\|_{\mathcal{D}}+n_*r\|\phi\|_{\mathcal{D}}+\|h_q\|_{L^1}+n_*qb+\sum_{k=1}^{m}c_k:=\ell.$$

Step 3. *N maps bounded sets into equicontinuous sets of Ω.*

Let $l_1,l_2\in[0,b]$, $l_1<l_2$, B_q be a bounded set of Ω as in Step 2, and let $y\in B_q$. Then

$$\left|N(y)(l_2)-N(y)(l_1)\right|\leqslant\int_{l_1}^{l_2}h_q(s)\,ds+n_*|l_2-l_1|+\sum_{0<t<l_2-l_1}c_k.$$

As $l_2\to l_1$ the right-hand side of the above inequality tends to zero. The equicontinuity for the cases $l_1<l_2\leqslant 0$ and $l_1\leqslant 0\leqslant l_2$ is obvious.

As a consequence of Steps 1 to 3 together with the Arzela–Ascoli theorem we can conclude that $N:\Omega\to\Omega$ is completely continuous.

Step 4. *Now it remains to show that the set*

$$\mathcal{E}(N):=\left\{y\in\Omega\colon\ y=\lambda N(y)\ \textit{for some}\ 0<\lambda<1\right\}$$

is bounded.

Let $y\in\mathcal{E}(N)$. Then $y=\lambda N(y)$ for some $0<\lambda<1$. Thus for each $t\in[0,b]$

$$y(t)=\lambda\left(\phi(0)+\sum_{i=1}^{n_*}\int_{-T_i}^{0}\phi(s)\,ds+\int_0^t f(s,y_s)\,ds\right.$$
$$\left.+\sum_{i=1}^{n_*}\int_0^{t-T_i}y(s)\,ds+\sum_{0<t_k<t}I_k\left(y\left(t_k^-\right)\right)\right).$$

This implies by (H2) and (H3) that for each $t\in J$ we have

$$\left|y(t)\right|\leqslant\|\phi\|_{\mathcal{D}}+n_*r\|\phi\|_{\mathcal{D}}+\sum_{k=1}^{m}c_k+\int_0^t p(s)\psi\left(\|y_s\|_{\mathcal{D}}\right)ds+n_*\int_0^t\left|y(s)\right|ds.$$

We consider the function μ defined by

$$\mu(t)=\sup\left\{\left|y(s)\right|\colon\ -r\leqslant s\leqslant t\right\},\quad 0\leqslant t\leqslant b.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in [0, b]$, by the previous inequality we have for $t \in [0, b]$

$$\mu(t) \leqslant \|\phi\|_{\mathcal{D}} + n_* r\|\phi\|_{\mathcal{D}} + \sum_{k=1}^{m} c_k + \int_0^t p(s)\psi\big(\mu(s)\big)\,ds + n_* \int_0^t \mu(s)\,ds.$$

If $t^* \in [-r, 0]$, then $\mu(t) = \|\phi\|_{\mathcal{D}}$ and the previous inequality holds. Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$c = v(0) = \|\phi\|_{\mathcal{D}} + n_* r\|\phi\|_{\mathcal{D}} + \sum_{k=1}^{m} c_k, \qquad \mu(t) \leqslant v(t), \quad t \in [0, b],$$

and

$$v'(t) = n_*\mu(t) + p(t)\psi\big(\mu(t)\big) \quad \text{a.e. } t \in [0, b].$$

Using the nondecreasing character of ψ we get

$$v'(t) \leqslant n_* v(t) + p(t)\psi\big(v(t)\big) \leqslant m(t)\big[v(t) + \psi\big(v(t)\big)\big] \quad \text{a.e. } t \in [0, b].$$

This implies that for each $t \in [0, b]$

$$\int_{v(0)}^{v(t)} \frac{ds}{s + \psi(s)} \leqslant \int_0^b m(s)\,ds < \int_{v(0)}^{\infty} \frac{ds}{s + \psi(s)}.$$

Thus there exists a constant K such that $v(t) \leqslant K$, $t \in [0, b]$, and hence $\mu(t) \leqslant K$, $t \in [0, b]$. Since for every $t \in [0, b]$, $\|y_t\| \leqslant \mu(t)$, we have

$$\|y\|_{\Omega} \leqslant \max\big\{\|\phi\|_{\mathcal{D}}, K\big\} := K',$$

where K' depends on b and on the functions m and ψ. This shows that $\mathcal{E}(N)$ is bounded.

Set $X := \Omega$. As a consequence of Schaefer theorem (see [15, p. 29]) we deduce that N has a fixed point y which is a solution to the problem (1)–(3). □

Remark 3.4. We can easily show that the above reasoning with appropriateness can be applied to obtain existence and uniqueness results for the first order impulsive neutral functional differential equation

$$\frac{d}{dt}\big[y(t) - g(t, y_t)\big] = f(t, y_t) + \sum_{i=1}^{n_*} y(t - T_i) \quad \text{a.e. } t \in J := [0, b]\backslash\{t_1, t_2, \dots, t_m\}, \tag{8}$$

$$y\big(t_k^+\big) - y(t_k) = I_k\big(y(t_k)\big), \quad k = 1, \dots, m, \tag{9}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \tag{10}$$

where r, f, I_k are as in the problem (1)–(3) and $g : J \times \mathcal{D} \to \mathbb{R}^n$.

In this part we given uniqueness result of the problem (1)–(3).

(A1) There exists $l \in L^1([0, b], \mathbb{R}_+)$ such that

$$\big|f(t, x) - f(t, \bar{x})\big| \leqslant l(t)\|x - \bar{x}\|_{\mathcal{D}} \quad \text{for all } x, \bar{x} \in \mathcal{D} \text{ and } t \in J;$$

(A2) There exist constants $\bar{c}_k \geqslant 0$, $k = 1, 2, \ldots, m$ such that

$$\left|I_k(y) - I_k(x)\right| \leqslant \bar{c}_k |x - \bar{x}| \quad \text{for each } x, \bar{x} \in \mathbb{R}^n.$$

Theorem 3.5. *Assume that hypotheses* (A1)–(A2) *hold. If* $\sum_{k=1}^{m} \bar{c}_k < 1$. *Then the IVP* (1)–(3) *has unique solutions.*

Proof. Let $N : \Omega \to \Omega$ be defined as in Theorem 3.3. We shall show that N is a contraction. Indeed, consider $y, \bar{y} \in \Omega$. Then we have for each $t \in [0, b]$

$$\begin{aligned}
\left|N(y)(t) - N(\bar{y})(t)\right| &\leqslant \int_0^t \left|f(s, y_s) - f(s, \bar{y}_s)\right| ds + \sum_{k=1}^{n_*} \int_0^{t-T_i} \left|y(s) - \bar{y}(s)\right| ds \\
&\quad + \sum_{0<t_k<t} \left|I_k\big(y\big(t_k^-\big)\big) - I_k\big(\bar{y}\big(t_k^-\big)\big)\right| \\
&\leqslant \int_0^t l(s) \|y_s - \bar{y}_s\|_{\mathcal{D}}\, ds + n_* \int_0^t \left|y(s) - \bar{y}(s)\right| ds \\
&\quad + \sum_{0<t_k<t} \bar{c}_k \left|y(t_k) - \bar{y}(t_k)\right| \\
&\leqslant \int_0^t l(s) e^{\tau L(s)} e^{-\tau L(s)} \|y_s - \bar{y}_s\|_{\mathcal{D}}\, ds \\
&\quad + \int_0^t n_* e^{\tau L(s)} e^{-\tau L(s)} \left|y(s) - \bar{y}(s)\right| ds \\
&\quad + \sum_{0<t_k<t} \bar{c}_k e^{\tau L(t)} e^{-\tau L(t)} \left|y(t_k) - \bar{y}(t_k)\right| \\
&\leqslant \int_0^t l(s) e^{\tau L(s)}\, ds \|y - \bar{y}\|_{B\Omega}\, ds + \int_0^t n_* e^{\tau L(s)}\, ds \|y - \bar{y}\|_{B\Omega} \\
&\quad + \sum_{0<t_k<t} \bar{c}_k e^{\tau L(t)} \|y - \bar{y}\|_{B\Omega} \\
&\leqslant \int_0^t \frac{1}{\tau} \big(e^{\tau L(s)}\big)'\, ds \|y - \bar{y}\|_{B\Omega}\, ds + \int_0^t \frac{1}{\tau} \big(e^{\tau L(s)}\big)'\, ds \|y - \bar{y}\|_{B\Omega} \\
&\quad + \sum_{k=1}^{m} \bar{c}_k e^{\tau L(t)} \|y - \bar{y}\|_{B\Omega} \\
&\leqslant e^{\tau L(t)} \left(\frac{2}{\tau} + \sum_{k=1}^{m} \bar{c}_k \right) \|y - \bar{y}\|_{B\Omega}.
\end{aligned}$$

Thus

$$e^{-\tau L(t)}\left|N(y)(t)-N(\bar{y})(t)\right| \leqslant \left(\frac{2}{\tau}+\sum_{k=1}^{m}\bar{c}_k\right)\|y-\bar{y}\|_{B\Omega},$$

where $L(t)=\int_{-r}^{t} l_*(s)\,ds$ and

$$l_*(t)=\begin{cases}0, & t\in[-r,0],\\ l(t)+n_*, & t\in[0,b]\end{cases}$$

and τ is sufficiently large and $\|\cdot\|_{B\Omega}$ is the Bielecki-type norm on Ω defined by

$$\|y\|_{B\Omega}=\sup_{t\in[-r,b]} e^{-\tau L(t)}\left|y(t)\right|.$$

Therefore,

$$\left\|N(y)-N(\bar{y})\right\|_{B\Omega} \leqslant \left(\frac{2}{\tau}+\sum_{k=1}^{m}\bar{c}_k\right)\|y-\bar{y}\|_{B\Omega},$$

showing that, N is a contraction and hence it has a unique fixed point which is a solution to (1)–(3). The proof is completed. □

4. Global existence and uniqueness result

In this section, we are concerned with an application of a recent nonlinear alternative for contraction maps in Fréchet spaces due to, Frigon and Granas [7], to the existence and uniqueness of the following problem, with infinity impulses and multiple delay

$$y'(t)=f(t,y_t)+\sum_{i=1}^{n_*} y(t-T_i) \quad \text{a.e. } t\in J_*:=[0,\infty)\setminus\{t_1,t_2,\dots\}, \tag{11}$$

$$y\big(t_k^+\big)-y\big(t_k^-\big)=I_k\big(y\big(t_k^-\big)\big), \quad k=1,\dots, \tag{12}$$

$$y(t)=\phi(t), \quad t\in[-r,0], \tag{13}$$

where $f: J_*\times\mathcal{D}\to\mathbb{R}^n$ and $I_k\in C(\mathbb{R}^n,\mathbb{R}^n)$, $k=1,\dots$, $\mathcal{D}=\{\psi:[-r,0]\to\mathbb{R}^n$, ψ is continuous everywhere except for a countable number of points $\bar{t}$ at which $\psi(\bar{t}^-)$ and $\psi(\bar{t}^+)$ exist, $\psi(\bar{t}^-)=\psi(\bar{t})$ and $\sup_{\theta\in[-r,0]}|\psi(\theta)|<\infty\}$ $(0<r<\infty)$, $0=t_0<t_1<\cdots<t_m<\cdots$, $\lim_{n\to\infty} t_n=\infty$.

As we know, the investigation of many properties of solutions for a given equation, such as stability, oscillation, needs its guarantee of global existence. Thus it is important and necessary to establish sufficient conditions for global existence of solutions for impulsive differential equations. The global existence results for impulsive differential equations with different conditions were studied by Benchohra et al. [4], Cheng and Yan [5], Graef and Ouahab [8], Guo [9,10], Guo and Liu [11], Marino et al. [13], Stamov and Stamova [16], Weng [17], Yan [18,19]. Very, recently this alternative was applied by Arara et al. [1] for controllability of functional semi-linear differential equations and by Graef and Ouahab [8] for functional impulsive differential equations with variable times. These results can be seen as a contribution to the literature.

For more details on the following notions we refer to [7]. Let X be a Fréchet space with a family of semi-norms $\{\|\cdot\|_n),\ n\in\mathbb{N}\}$. Let $Y\subset X$, we say that Y is bounded if for every $n\in\mathbb{N}$, there exists $M_n>0$ such that

$$\|y\|_n\leqslant M_n \quad \text{for all } y\in Y.$$

To X, we associate a sequence of Banach spaces $\{(X^n, \|\cdot\|_n)\}$ as follows. For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by $x \sim_n y$ if and only if $\|x - y\|_n = 0$ for all $x, y \in X$. We denote $X^n = (X^/ \sim_n, \|\cdot\|)$ the quotient space, the completion of X^n with respect to $\|\cdot\|_n$. To every $Y \subset X$, we associate a sequence the $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows. For every $x \in X$, we denote $[x]_n$ the equivalence class of x of subset X^n and we defined $Y^n = \{[x]_n\colon x \in Y\}$. We denote $\overline{Y}^n$, $\mathrm{int}_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\|\cdot\|$ in X^n. We assume that the family of semi-norms $\{\|\cdot\|_n\}$ verifies

$$\|x\|_1 \leqslant \|x\|_2 \leqslant \|x\|_3 \leqslant \cdots \quad \text{for every } x \in X.$$

Definition 4.1. A function $f : X \to X$ is said to be a contraction if for each $n \in \mathbb{N}$ there exists $k_n \in (0, 1)$ such that

$$\|f(x) - f(y)\|_n \leqslant k_n\|x - y\|_n \quad \text{for all } x, y \in X.$$

Theorem 4.2 *(Nonlinear alternative [7]). Let X be a Fréchet space and $Y \subset X$ a closed subset in Y let $N : Y \to X$ be a contraction such that $N(Y)$ is bounded. Then one of the following statements hold*:

(C1) *N has a unique fixed point*;
(C2) *there exists $\lambda \in [0, 1)$, $n \in \mathbb{N}$, and $x \in \partial_n Y^n$ such that $\|x - \lambda N(x)\|_n = 0$.*

5. Uniqueness result in Fréchet spaces

In order to define the solution of (1)–(3) we shall consider the space, $PC(J, \mathbb{R}^n) = \{y : [0, \infty) \to \mathbb{R}^n$ such that $y(t)$ is continuous everywhere except for some t_k at which $y(t_k^-)$ and $y(t_k^+)$ exist and $y(t_k^-) = y(t_k),\ k = 1, 2, \ldots\}$.

Set

$$\Omega = \{y : J_1 \to \mathbb{R}^n\colon\ y \in \mathcal{D} \cap PC(J_*, \mathbb{R}^n)\}, \quad J_1 = [-r, 0] \cup J_*.$$

Definition 5.1. A function $y \in \Omega$ is said to be a solution of (11)–(13) if

$$y'(t) = f(s, y_t) + \sum_{k=1}^{n_*} y(t - T_i) \quad \text{a.e. } t \in [0, \infty),\ t \neq t_k,\ k = 1, 2, \ldots,$$

and the conditions (12) and (13) are satisfied.

Let us introduce the following hypotheses:

(B1) There exist a function $p \in L^1(J_*, \mathbb{R}_+)$ and a continuous nondecreasing function $\psi : [0, \infty) \to [0, \infty)$ such that

$$\|f(t, x)\| \leqslant p(t)\psi(\|x\|_{\mathcal{D}}) \quad \text{for a.e. } t \in J_* \text{ and each } x \in \mathcal{D},$$

with

$$\int_1^\infty \frac{ds}{s + \psi(s)} = \infty;$$

(B2) There exist constants $c_k > 0$ such that

$$\left|I_k(y)\right| \leqslant c_k \quad \text{for each } k = 1, \dots, \text{ and for all } y \in \mathbb{R}^n;$$

(B3) For all $R > 0$ there exists $l_R \in L^1_{\text{loc}}([0,\infty), \mathbb{R}_+)$ such that

$$\left|f(t,x) - f(t,\bar{x})\right| \leqslant l_R(t)\|x - \bar{x}\|_{\mathcal{D}} \quad \text{for all } x, \bar{x} \in \mathcal{D} \text{ with } \|x\|, \|\bar{x}\| \leqslant R, \text{ for a.e. } t \in J_*;$$

(B4) There exist constants $\bar{c}_k \geqslant 0$ such that for each $k = 1, 2, \dots,$ we have

$$\left|I_k(y) - I_k(x)\right| \leqslant \bar{c}_k|x - \bar{x}| \quad \text{for each } x, \bar{x} \in \mathbb{R}^n.$$

Theorem 5.2. *Assume that hypotheses* (B1)–(B4) *hold. If* $\sum_{k=1}^{\infty} \bar{c}_k < 1$*, then the IVP* (11)–(13) *has unique solutions.*

Proof. We begin by defining a family of semi-norms on Ω, thus rendering Ω into a Fréchet space. Let τ be sufficiently large. Then for each $n \in \mathbb{N}$ we define in Ω the semi-norms by

$$\|y\|_n = \sup\left\{e^{-\tau L_n(t)}\left|y(t)\right|: -r \leqslant t \leqslant t_n\right\},$$

where $L_n(t) = \int_{-r}^{t} \bar{l}_n(s)\,ds$ and

$$\bar{l}_n(t) = \begin{cases} 0, & t \in [-r, 0], \\ l_n(t) + n_*, & t \in [0, t_n]. \end{cases}$$

Thus $\Omega = \bigcup_{n \geqslant 1} \Omega_n$, where

$$\Omega_n = \left\{y : [-r, t_n] \to \mathbb{R}^n:\ y \in \mathcal{D} \cap PC_n\left(J, \mathbb{R}^n\right)\right\}$$

and $PC_n(J, \mathbb{R}^n) = \{y : [0, t_n] \to \mathbb{R}^n$ such that $y(t)$ is continuous everywhere except for some t_k at which $y(t_k^-)$ and $y(t_k^+)$ exist and $y(t_k^-) = y(t_k)$, $k = 1, 2, \dots, n-1\}$. Then Ω is a Fréchet space with the family of semi-norms $\{\|\cdot\|_n\}$.

Transform the problem (11)–(13) into a fixed point problem. Consider the operator $G : \Omega \to \Omega$ defined by

$$G(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [-r, 0], \\ \phi(0) + \sum_{i=1}^{n_*} \int_{-T_i}^{0} \phi(s)\,ds + \int_0^t f(s, y_s)\,ds & \\ \quad + \sum_{i=1}^{n_*} \int_0^{t-T_i} y(s)\,ds + \sum_{0<t_k<t} I_k(y(t_k^-)) & \text{if } t \in [0, \infty). \end{cases}$$

Let y be a solutions of the problem (11)–(13) then for $t \in [0, t_n]$, $n \in \mathbb{N}$ we have

$$y(t) = \phi(0) + \sum_{i=1}^{n_*} \int_{-T_i}^{0} \phi(s)\,ds + \int_0^t f(s, y_s)\,ds + \sum_{i=1}^{n_*} \int_0^{t-T_i} y(s)\,ds + \sum_{0<t_k<t} I_k\left(y\left(t_k^-\right)\right).$$

As in Theorem 3.3 we can prove that there exists $M_n > 0$ such that $\|y\|_n \leqslant M_n$. Set

$$Y = \left\{y \in \Omega:\ \|y\|_n \leqslant M_n + 1 \text{ for all } n \in \mathbb{N}\right\}.$$

Clearly, Y is a closed subset of Ω and we can easily show that $G : \Omega_n \to \Omega_n$ is contraction. From the choice of Y there is no $y \in \partial Y^n$ such that $y = \lambda G(y)$ for some $\lambda \in (0, 1)$. As a consequence of the nonlinear alternative type (see Theorem 4.2) we deduce that G has a unique fixed point which is a solution to (11)–(13). $\square$

Remark 5.3. We can replace (H3) by the weaker condition:

(H*3) There exist a continuous function $\psi:[0,\infty)\to(0,\infty)$ and $p\in L^1([0,b],\mathbb{R}^n)$ such that

$$\left|f(t,y)\right|\leqslant p(t)\psi\left(\|y\|_{\mathcal{D}}\right)\quad\text{for a.e. } t\in[0,b] \text{ and each } y\in\mathcal{D}$$

with

$$\int_{t_{k-1}}^{t_k} m(t)\,ds<\int_{\overline{N}_{k-1}}^{\infty}\frac{du}{u+\psi(u)},\qquad k=1,\dots,m,$$

where

$$\overline{N}_0=\|\phi\|_{\mathcal{D}}+\sum_{i=1}^{n_*}T_i\|\phi\|_{\mathcal{D}},\qquad N_{k-1}=\sup_{x\in[-K_{k-2},K_{k-2}]}\left|I_{k-1}(x)\right|+M_{k-2},$$

$$\overline{N}_{k-1}=N_{k-1}+n_*t_{k-1}K_{k-2},$$

$$M_{k-2}=\Gamma_{k-1}^{-1}\left(\int_{t_{k-2}}^{t_{k-1}}m(s)\,ds\right)\quad\text{for } k=2,\dots,m+2,\quad\text{and}$$

$$K_0=\max\left(M_0,\|\phi\|_{\mathcal{D}}\right),\qquad K_k=\max(K_{k-1},M_k),\quad k=1,\dots,m+1,$$

$$m(t)=\max\left(p(t),n_*\right),$$

$$\Gamma_{l-1}(z)=\int_{\overline{N}_{l-1}}^{z}\frac{du}{u+\psi(u)},\qquad z\geqslant\overline{N}_{l-1},\ l\in\{1,\dots,m+2\}.$$

Then for each $k=0,\dots,m+1$ there exists a constant K_k such that

$$\sup\left\{\left|y(t)\right|:\ t\in[t_{k-1},t_k]\right\}\leqslant K_k,$$

for each solution y of the problem (1)–(3).

Proof. Let y be a possible solution of the problem (1)–(3). Then $y|_{[-r,t_1]}$ is a solution to

$$y'(t)=f(t,y_t)+\sum_{i=1}^{n_*}y(t-T_i)\quad\text{for } t\in[0,t_1],\qquad y(t)=\phi(t),\quad t\in[-r,0].$$

By integration of above equations, we get

$$y(t)=\phi(0)+\int_0^t f(s,y_s)\,ds+\sum_{i=1}^{n_*}\int_{-T_i}^{0}\phi(s)\,ds+\sum_{i=1}^{n_*}\int_0^t y(s)\,ds.$$

Hence by (H*3) we have

$$\left|y(t)\right|\leqslant\|\phi\|_{\mathcal{D}}+\sum_{i=1}^{n_*}T_i\|\phi\|_{\mathcal{D}}+\int_0^t p(s)\psi\left(\|y_s\|_{\mathcal{D}}\right)ds+n_*\int_0^t\left|y(s)\right|ds.$$

We consider the function μ defined by

$$\mu(t)=\sup\{|y(s)|:\ -r\leqslant s\leqslant t\},\quad 0\leqslant t\leqslant t_1.$$

Let $t^*\in[-r,t]$ be such that $\mu(t)=|y(t^*)|$. If $t^*\in[0,t_1]$, by the previous inequality we have for $t\in[0,t_1]$

$$\mu(t)\leqslant\|\phi\|_{\mathcal{D}}+\sum_{i=1}^{n_*}T_i\|\phi\|_{\mathcal{D}}+\int_0^t m(s)\big[\mu(s)+\psi\big(\mu(s)\big)\big]\,ds.$$

If $t^*\in[-r,0]$, then $\mu(t)=\|\phi\|_{\mathcal{D}}$ and the previous inequality holds. Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$c=v(0)=\|\phi\|_{\mathcal{D}}+\sum_{i=1}^{n_*}T_i\|\phi\|_{\mathcal{D}},\qquad \mu(t)\leqslant v(t),\quad t\in[0,t_1],$$

and

$$v'(t)\leqslant m(t)\big[\mu(t)+\psi\big(\mu(t)\big)\big]\quad \text{a.e. } t\in[0,t_1].$$

Using the nondecreasing character of ψ we get

$$v'(t)\leqslant m(t)\big[v(t)+\psi\big(v(t)\big)\big]\quad \text{a.e. } t\in[0,t_1].$$

This implies that for each $t\in[0,t_1]$

$$\Gamma_1\big(v(t)\big)=\int_{v(0)}^{v(t)}\frac{ds}{s+\psi(s)}\leqslant\int_0^{t_1}m(s)\,ds<\int_{v(0)}^{\infty}\frac{ds}{s+\psi(s)}.$$

Thus there exists a constant K such that $v(t)\leqslant\Gamma_1^{-1}(\int_0^{t_1}m(s)\,ds)=M_0$, $t\in[0,t_1]$, and hence $\mu(t)\leqslant M_0$, $t\in[0,t_1]$. Since for every $t\in[0,t_1]$, $\|y_t\|\leqslant\mu(t)$, we have

$$\|y\|_\infty\leqslant\max\{\|\phi\|_{\mathcal{D}},M_0\}=K_0.$$

Now $y|_{[t_1,t_2]}$ is a solution to

$$y'(t)=f(t,y_t)+\sum_{i=1}^{n_*}y(t-T_i)\quad \text{a.e. } t\in[t_1,t_2],\qquad y\big(t_1^+\big)-y\big(t_1^-\big)=I_1\big(y\big(t_1^-\big)\big).$$

Note that

$$\big|y\big(t_1^+\big)\big|\leqslant\sup_{x\in[-K_0,K_0]}\big|I_1(x)\big|+K_0:=N_1.$$

Next

$$\big|y(t)\big|\leqslant N_1+\sum_{i=1}^{n_*}\int_0^{t_1}\big|y(s)\big|\,ds+\int_{t_1}^t p(s)\psi\big(\|y_s\|_{\mathcal{D}}\big)\,ds+\sum_{i=1}^{n_*}\int_{t_1}^t\big|y(s)\big|\,ds.$$

We consider the function μ defined by

$$\mu(t)=\sup\{|y(s)|:\ -r\leqslant s\leqslant t\},\quad 0\leqslant t\leqslant t_2.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in [0, t_2]$, by the previous inequality we have for $t \in [0, t_2]$

$$\mu(t) \leqslant N_1 + n_* t_1 K_0 + \int_{t_1}^{t} m(s)\big[\mu(s) + \psi\big(\mu(s)\big)\big]\,ds.$$

If $t^* \in [-r, t_1]$, then $\mu(t) \leqslant K_0$ and the previous inequality holds. Let us take the right-hand side of the above inequality as $w(t)$. Then we have

$$c = v(t_1) = N_1 + t_1 K_0 n_*, \qquad \mu(t) \leqslant w(t), \quad t \in [t_1, t_2],$$

and

$$w'(t) = m(t)\big[\mu(t) + \psi\big(\mu(t)\big)\big] \quad \text{a.e. } t \in [t_1, t_2].$$

Using the nondecreasing character of ψ we get

$$w'(t) \leqslant m(t)\big[w(t) + \psi\big(w(t)\big)\big] \quad \text{a.e. } t \in [t_1, t_2].$$

This implies that for each $t \in [t_1, t_2]$

$$\Gamma_2\big(w(t)\big) = \int_{w(t_1)}^{w(t)} \frac{ds}{s + \psi(s)} \leqslant \int_{t_1}^{t_2} m(s)\,ds < \int_{w(t_1)}^{\infty} \frac{ds}{s + \psi(s)}.$$

Thus there exists a constant K such that $v(t) \leqslant \Gamma_2^{-1}(\int_{t_1}^{t_2} m(s)\,ds) = M_1$, $t \in [t_1, t_2]$, and hence $\mu(t) \leqslant M_1$, $t \in [t_1, t_2]$. Since for every $t \in [t_1, t_2]$, $\|y_t\| \leqslant \mu(t)$, we have

$$\|y\|_\infty \leqslant \max\{K_0, M_1\} = K_1.$$

We continue this process and also take into account that $y|_{[t_m, b]}$ solution to the problem

$$y'(t) = f(t, y_t) + \sum_{i=1}^{n_*} y(t - T_i) \quad \text{a.e. } t \in [t_m, b], \qquad y\big(t_{m-1}^+\big) - y\big(t_m^-\big) = I_m\big(y\big(t_m^-\big)\big).$$

We obtain that there exists a constant M_{m+1} such that

$$\sup\big\{\big|y(t)\big|\colon t \in [t_m, b]\big\} \leqslant \max(M_{m+1}, K_{m-1}) := K_{m+1},$$
$$\text{where } M_{m+1} = \Gamma_{m+1}^{-1}\left(\int_{t_m}^{b} m(s)\,ds\right).$$

Consequently, for each possible solution y to (1)–(3) we have

$$\|y\|_\Omega \leqslant \max\big\{\|\phi\|_{\mathcal{D}},\ K_i, i = 1, \ldots, m\big\} := \overline{M}. \qquad \square$$

Theorem 5.4. *Assume that* (H1) *and* (H*3) *are satisfied. Then the problem* (1)–(3) *has at least one solution.*

Proof. Consider the operator N defined in the proof of Theorem 3.3. We shall show that N satisfies the assumptions of the nonlinear alternative of Leray–Schauder type. Set

$$U = \big\{y \in \Omega\colon \|y\|_\Omega < \overline{M} + 1\big\}.$$

As in Theorem 3.3 the operator $N : \overline{U} \to \Omega$ is continuous and completely continuous. From the choice of U there is no $y \in \partial U$ such that $y = \lambda N(y)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray–Schauder type [6], we deduce that N has a fixed point y in U, which is a solution of the IVP (1)–(3). □

6. Some examples

In this, section we give some examples to illustrate the usefulness of our main results.

Example 6.1. Consider the system

$$y'(t) = \frac{1}{(t+1)(t+2)} y_t^2 + y(t-1) \quad \text{a.e. } t \in J := [0,1] \setminus \left\{\frac{1}{2}\right\}, \tag{14}$$

$$y\left(\frac{1}{2}^+\right) - y\left(\frac{1}{2}^-\right) = by\left(\frac{1}{2}^-\right), \tag{15}$$

$$y(t) = \phi(t), \quad t \in [-1,0], \tag{16}$$

where

$$\phi(t) = \begin{cases} 0, & \text{if } t = 0, \\ t - \frac{1}{2}, & \text{if } t \in [-1,0). \end{cases}$$

$I_1(x) = bx$ and $f(t,x) = \frac{1}{(t+1)(t+2)} x^2$. Assume that $p(t) = \frac{1}{t+1}$ and $\psi(x) = x^2 + 1$. Then

$$\left|f(t,x)\right| \leqslant \frac{1}{t+2} \psi\left(\|x\|_{\mathcal{D}}\right) \quad \text{for all } x \in \mathcal{D},\ t \in [0,1],$$

$$\int_0^1 m(t)\,dt = 1 < \int_0^\infty \frac{du}{u^2+u+1} = \frac{\pi}{2\sqrt{2}}.$$

Thus, by Theorem 5.4 the problem (14)–(16) has at least one solution.

Example 6.2. Consider the system

$$y'(t) = \frac{1}{(t+1)(t+2)} y_t^2 + y(t-1) \quad \text{a.e. } t \in J := [0,\infty)] \setminus \{t_1, t_2, \ldots\}, \tag{17}$$

$$y\left(t_k^+\right) - y\left(t_k^-\right) = b_k y\left(t_k^-\right), \quad k = 1, \ldots, m, \tag{18}$$

$$y(t) = \phi(t), \quad t \in [-1,0], \tag{19}$$

where $t_k = (6k^2 + 12k - 2)/6k$, $k \in \mathbb{N}$, and

$$\phi(t) = \begin{cases} 0, & \text{if } t = 0, \\ t - \frac{1}{2}, & \text{if } t \in [-1,0). \end{cases}$$

$f(t,x) = \frac{1}{(t+1)(t+2)} x^2$ and $b_k > 0$, $I_k(x) = b_k x$, $k \in \mathbb{N}$. Let $R > 0$ and $x, \bar{x} \in \mathcal{D}$, such that $\|x\|_{\mathcal{D}}, \|\bar{x}\|_{\mathcal{D}} \leqslant R$ hence

$$\left|f(t,x) - f(t,\bar{x})\right| \leqslant \frac{1}{(t+1)(t+2)} \|x + \bar{x}\|_{\mathcal{D}} \|x - \bar{x}\|_{\mathcal{D}} \leqslant \frac{2R}{(t+1)(t+2)} \|x - \bar{x}\|_{\mathcal{D}}.$$

Set $l_R(t) = \frac{2R}{(t+1)(t+2)}$ for $t \in [0,\infty) \Rightarrow l_R \in L^1_{\mathrm{loc}}([0,\infty),\mathbb{R}^n)$. It is clearly that

$$\left|I_k(x) - I_k(\bar{x})\right| \leqslant b_k|x-\bar{x}| \quad \text{for all } x, \bar{x} \in \mathbb{R}^n.$$

If $\sum_{k=1}^{\infty} b_k < 1$. Then by Theorem 5.2 the problem (17)–(18) has unique solution.

Acknowledgment

The author thanks the referee for his valuable suggestions and comments.

References

[1] A. Arara, M. Benchohra, A. Ouahab, Some uniqueness results for the controllability of functional semilinear differential equations in Fréchet space, Nonlinear Oscil. 6 (3) (2003) 287–303.
[2] D.D. Bainov, P.S. Simeonov, Systems with Impulse Effect, Horwood, Chichester, 1989.
[3] M. Benchohra, J. Henderson, S.K. Ntouyas, An existence result for first order impulsive functional differential equations in Banach spaces, Comput. Math. Appl. 42 (10–11) (2001) 1303–1310.
[4] M. Benchohra, J. Henderson, S.K. Ntouyas, A. Ouahab, Boundary value problems for impulsive functional differential equations with infinite delay, Int. J. Math. Comput. Sci. 1 (2006) 27–39.
[5] D. Cheng, J. Yan, Global existence and asymptotic behavior of solution of second order nonlinear impulsive differential equations, Int. J. Math. Math. Sci. 25 (2001) 175–182.
[6] J. Dugundji, A. Granas, Fixed Point Theory, Mongrafie Mat., PWN, Warsaw, 1982.
[7] M. Frigon, A. Granas, Résultats de type Leray–Schauder pour des contractions sur des espaces de Fréchet, Ann. Sci. Math. Québec 22 (2) (1998) 161–168.
[8] J.R. Graef, A. Ouahab, Some existence results and uniqueness solutions for functional impulsive differential equations with variable times in Fréchet spaces, Dyn. Contin. Discrete Impuls. Syst., in press.
[9] D. Guo, Boundary value problem for impulsive integro-differential equation on unbounded domains in a Banach space, Appl. Math. Comput. 99 (1999) 1–15.
[10] G. Guo, A class of second-order implusive integro-differential equations on unbounded domain in a Banach space, Appl. Math. Comput. 125 (2002) 59–77.
[11] D. Guo, X.Z. Liu, Impulsive integro-differential equations on unbounded domain in Banach space, Nonlinear Stud. 3 (1996) 49–57.
[12] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.
[13] G. Marino, P. Pielramala, L. Muglia, Impulsive neutral integrodifferential equations on unbounded intervals, Mediterr. J. Math. 1 (2004) 3–42.
[14] A.M. Samoilenko, N.A. Perestyuk, Impulsive Differential Equations, World Scientific, Singapore, 1995.
[15] D.R. Smart, Fixed Point Theorems, Cambridge Univ. Press, Cambridge, 1974.
[16] G.T. Stamov, I.M. Stamova, Second method of Lyapunov and existence of integral manifolds for impulsive differential equations, SUT J. Math. 32 (1996) 101–107.
[17] P. Weng, Global existence in integrable space for impulsive FDE with P-delay, Dyn. Contin. Discrete Impuls. Syst. 9 (2002) 321–337.
[18] B. Yan, On L^p_{loc}-solutions of nonlinear impulsive Volterra integral equations in Banach spaces, SUT J. Math. 33 (1997) 121–137.
[19] B. Yan, The existence of positive solutions of nonlinear impulsive Fredholm integral equations in Banach spaces, Dyn. Contin. Discrete Impuls. Syst. 6 (1999) 289–300.